

三次元球面座標系におけるラプラシアンを表式を求めるラプラシアンは関数 f に対して

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

であるが、これを変換して直交曲線座標系 (r, θ, ϕ) について求める。<https://hogeocraft.blogspot.com/2016/12/python-sympy.html> (<https://hogeocraft.blogspot.com/2016/12/python-sympy.html>) を参考にした。

```
In [1]: from sympy import *
init_printing()
r = Symbol('r', real=True, positive=True)
theta = Symbol('theta', real=True, positive=True, domain=Interval(0, pi))
phi = Symbol('\phi', real=True, positive=True, domain=Interval(0, 2*pi))
```

極座標を以下で定義する

```
In [2]: x = r * sin(theta) * cos(phi)
y = r * sin(theta) * sin(phi)
z = r * cos(theta)
```

```
In [3]: xyz = [x, y, z]
xyz
```

```
Out[3]: [r sin(theta) cos(phi), r sin(theta) sin(phi), r cos(theta)]
```

```
In [4]: rtp = [r, theta, phi]
rtp
```

```
Out[4]: [r, theta, phi]
```

$(\frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \phi})^t = J(\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z})^t$ を求める

```
In [5]: J = Matrix(3, 3, [Derivative(c1, c2) for c2 in rtp for c1 in xyz])
J = J.doit()
J
```

```
Out[5]: Matrix([
[ sin(theta) cos(phi), sin(phi) sin(theta), cos(theta) ],
[ r cos(phi) cos(theta), r sin(phi) cos(theta), -r sin(theta) ],
[ -r sin(phi) sin(theta), r sin(theta) cos(phi), 0 ]])
```

J の逆行列を求めることが重要となる。

```
In [6]: InvJ = simplify(J.inv())
InvJ
```

```
Out[6]: Matrix([
[ sin(theta) cos(phi), cos(phi) cos(theta)/r, -sin(phi)/r sin(theta) ],
[ sin(phi) sin(theta), sin(phi) cos(theta)/r, cos(phi)/r sin(theta) ],
[ cos(theta), -sin(theta)/r, 0 ]])
```

Jの逆行列演算子 = ヤコビ行列を二回作用させて二回偏微分を得る。

```
In [7]: f = Function('f')(r, theta, phi)
```

```
In [8]: a = Matrix(3 , 1 , [Derivative(f , c) for c in rtp])
a
```

Out[8]:

$$\begin{bmatrix} \frac{\partial}{\partial r} f(r, \theta, \phi) \\ \frac{\partial}{\partial \theta} f(r, \theta, \phi) \\ \frac{\partial}{\partial \phi} f(r, \theta, \phi) \end{bmatrix}$$

```
In [9]: gradf = InvJ*a
gradf
```

Out[9]:

$$\begin{bmatrix} \sin(\theta) \cos(\phi) \frac{\partial}{\partial r} f(r, \theta, \phi) - \frac{\sin(\phi) \frac{\partial}{\partial \phi} f(r, \theta, \phi)}{r \sin(\theta)} + \frac{\cos(\phi) \cos(\theta) \frac{\partial}{\partial \theta} f(r, \theta, \phi)}{r} \\ \sin(\phi) \sin(\theta) \frac{\partial}{\partial r} f(r, \theta, \phi) + \frac{\sin(\phi) \cos(\theta) \frac{\partial}{\partial \theta} f(r, \theta, \phi)}{r} + \frac{\cos(\phi) \frac{\partial}{\partial \phi} f(r, \theta, \phi)}{r \sin(\theta)} \\ \cos(\theta) \frac{\partial}{\partial r} f(r, \theta, \phi) - \frac{\sin(\theta) \frac{\partial}{\partial \theta} f(r, \theta, \phi)}{r} \end{bmatrix}$$

```
In [14]: simplify(sum([ (InvJ * Matrix(3,1,[ Derivative(gradf[i],e) for e in rtp ])).doit()[i] for i in
range(3) ]))
```

Out[14]:

$$\frac{r^2 \frac{\partial^2}{\partial r^2} f(r, \theta, \phi) + 2r \frac{\partial}{\partial r} f(r, \theta, \phi) + \frac{\partial^2}{\partial \theta^2} f(r, \theta, \phi) + \frac{\frac{\partial}{\partial \theta} f(r, \theta, \phi)}{\tan(\theta)} + \frac{\frac{\partial^2}{\partial \phi^2} f(r, \theta, \phi)}{\sin^2(\theta)}}{r^2}$$

```
In [ ]:
```