三次元球面座標系におけるラプラシアンの表式を求めるラプラシアンは関数fに対して
$\nabla^2 f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
であるが、これを変換して直交曲線座標系(r, θ, ϕ)について求める。 [https://hogeocraft.blogspot.com/2016/12/python-sympy.html (https://hogeocraft.blogspot.com/2016/12/python-sympy.html)](https://hogeocraft.blogspot.com/2016/12/python-sympy.html) を参考にした。

In [1]:
```
from sympy import *
init_printing()
r = Symbol('r',real=True, positive=True)
θ  = Symbol('theta',real=True, positive=True,domain=Interval(0,pi))
ϕ = Symbol('\phi',real=True, positive=True,domain=Interval(0,2*pi))
```

極座標を以下で定義する

In [2]:
```
x = r * sin(θ) * cos(ϕ)
y = r * sin(θ) * sin(ϕ)
z = r * cos(θ)
```

In [3]:
```
xyz = [x, y, z]
xyz
```

Out[3]: $[r \sin(\theta) \cos(\phi), \quad r \sin(\phi) \sin(\theta), \quad r \cos(\theta)]$

In [4]:
```
rtp = [r, θ, ϕ]
rtp
```

Out[4]: $[r, \quad \theta, \quad \phi]$

$(\frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \phi})^t = J(\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z})^t$ を求める

In [5]:
```
J = Matrix(3 , 3 , [Derivative(c1 , c2) for c2 in rtp for c1 in xyz])
J = J.doit()
J
```

Out[5]:
$$\begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ r\cos(\phi)\cos(\theta) & r\sin(\phi)\cos(\theta) & -r\sin(\theta) \\ -r\sin(\phi)\sin(\theta) & r\sin(\theta)\cos(\phi) & 0 \end{bmatrix}$$

J の逆行列を求めることが重要となる。

In [6]:
```
InvJ = simplify(J.inv())
InvJ
```

Out[6]:
$$\begin{bmatrix} \sin(\theta)\cos(\phi) & \frac{\cos(\phi)\cos(\theta)}{r} & -\frac{\sin(\phi)}{r\sin(\theta)} \\ \sin(\phi)\sin(\theta) & \frac{\sin(\phi)\cos(\theta)}{r} & \frac{\cos(\phi)}{r\sin(\theta)} \\ \cos(\theta) & -\frac{\sin(\theta)}{r} & 0 \end{bmatrix}$$

Jの逆行列演算子 = ヤコビ行列を二回作用させて二回偏微分を得る。

In [7]:
```
f = Function('f')(r , θ , ϕ)
```

In [8]:
```
a = Matrix(3 , 1 , [Derivative(f , c) for c in rtp])
a
```

Out[8]:

$$\left[\begin{matrix}\frac{\partial}{\partial r} f{\left(r,\theta,\phi \right)}\\\frac{\partial}{\partial \theta} f{\left(r,\theta,\phi \right)}\\\frac{\partial}{\partial \phi} f{\left(r,\theta,\phi \right)}\end{matrix}\right]$$

In [9]:
```
gradf = InvJ*a
gradf
```

Out[9]:

$$\left[\begin{matrix}\sin{\left(\theta \right)} \cos{\left(\phi \right)} \frac{\partial}{\partial r} f{\left(r,\theta,\phi \right)} - \frac{\sin{\left(\phi \right)} \frac{\partial}{\partial \phi} f{\left(r,\theta,\phi \right)}}{r \sin{\left(\theta \right)}} + \frac{\cos{\left(\phi \right)} \cos{\left(\theta \right)} \frac{\partial}{\partial \theta} f{\left(r,\theta,\phi \right)}}{r}\\\sin{\left(\phi \right)} \sin{\left(\theta \right)} \frac{\partial}{\partial r} f{\left(r,\theta,\phi \right)} + \frac{\sin{\left(\phi \right)} \cos{\left(\theta \right)} \frac{\partial}{\partial \theta} f{\left(r,\theta,\phi \right)}}{r} + \frac{\cos{\left(\phi \right)} \frac{\partial}{\partial \phi} f{\left(r,\theta,\phi \right)}}{r \sin{\left(\theta \right)}}\\\cos{\left(\theta \right)} \frac{\partial}{\partial r} f{\left(r,\theta,\phi \right)} - \frac{\sin{\left(\theta \right)} \frac{\partial}{\partial \theta} f{\left(r,\theta,\phi \right)}}{r}\end{matrix}\right]$$

In [14]:
```
simplify(sum([ (InvJ * Matrix(3,1,[ Derivative(gradf[i],e) for e in rtp ]).doit())[i] for i in
range(3) ]))
```

Out[14]:

$$\frac{r^{2} \frac{\partial^{2}}{\partial r^{2}} f{\left(r,\theta,\phi \right)} + 2 r \frac{\partial}{\partial r} f{\left(r,\theta,\phi \right)} + \frac{\partial^{2}}{\partial \theta^{2}} f{\left(r,\theta,\phi \right)} + \frac{\frac{\partial}{\partial \theta} f{\left(r,\theta,\phi \right)}}{\tan{\left(\theta \right)}} + \frac{\frac{\partial^{2}}{\partial \phi^{2}} f{\left(r,\theta,\phi \right)}}{\sin^{2}{\left(\theta \right)}}}{r^{2}}$$

In [ ]:
```
```